

More cubic surfaces violating the Hasse principle

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Abstract

We generalize L. J. Mordell's construction [Mo] of cubic surfaces for which the Hasse principle fails.

Résumé

Nous généralisons la construction due à L. J. Mordell [Mo] des surfaces cubiques pour lesquelles la principe de Hasse est fausse.

1 Introduction and main result

1.1. — Sir P. Swinnerton-Dyer [SD] was the first to construct a cubic surface over \mathbb{Q} for which the Hasse principle provably fails. Swinnerton-Dyer's construction had soon been generalized by L. J. Mordell [Mo] who found two series of such examples. The starting points of Mordell's construction are the cubic number fields contained in $\mathbb{Q}(\zeta_p)$ for $p = 7$ and $p = 13$, respectively.

The failure of the Hasse principle in Mordell's examples may be explained more conceptually by the Brauer-Manin obstruction. This was observed by Yu. I. Manin in his book [Ma, Proposition 47.4].

1.2. — In this note, we will show that Mordell's construction may be generalized to an arbitrary prime $p \equiv 1 \pmod{3}$.

1.3. Notation. — i) We denote by K/\mathbb{Q} the unique cubic field extension contained in the cyclotomic extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$.

ii) We fix the explicit generator $\theta \in K$ given by $\theta := \text{tr}_{\mathbb{Q}(\zeta_p)/K}(\zeta_p - 1)$. More concretely, $\theta = -n + \sum_{i \in (\mathbb{F}_p^*)^3} \zeta_p^i$ for $n := \frac{p-1}{3}$.

1.4. Theorem. — Consider the cubic surface $X \subset \mathbf{P}_{\mathbb{Q}}^3$, given by

$$T_3(a_1T_0 + d_1T_3)(a_2T_0 + d_2T_3) = \prod_{i=1}^3 (T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2) .$$

Here, a_1, a_2, d_1, d_2 are integers and $\theta^{(i)}$ are the images of θ under $\text{Gal}(K/\mathbb{Q})$.

i) Then, the reduction X_p of X at p is given by $T_3(a_1T_0 + d_1T_3)(a_2T_0 + d_2T_3) = T_0^3$. Over the algebraic closure, X_p is the union of three planes. These are given by

$$T_3/T_0 = s_1, \quad T_3/T_0 = s_2, \quad T_3/T_0 = s_3$$

for s_i the zeroes of $T(a_1 + d_1T)(a_2 + d_2T) - 1 = 0$.

ii) Suppose $p \nmid d_1d_2$ and $\gcd(d_1, d_2) = 1$. Then, for every $(t_0 : t_1 : t_2 : t_3) \in X(\mathbb{Q})$, $s := (t_3/t_0 \bmod p)$ admits the property that

$$\frac{a_1 + d_1s}{s}$$

is a cube in \mathbb{F}_p^* .

In particular, if $\frac{a_1 + d_1s_i}{s_i} \in \mathbb{F}_p^*$ is a non-cube for every i such that $s_i \in \mathbb{F}_p$ then $X(\mathbb{Q}) = \emptyset$.

iii) Assume that $p \nmid d_1d_2$, that $\gcd(a_1, d_1)$ and $\gcd(a_2, d_2)$ contain only prime factors which decompose in K , and that $T(a_1 + d_1T)(a_2 + d_2T) - 1 = 0$ has at least one zero which is simple and in \mathbb{F}_p . Then, $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$.

1.5. — For $p = 7$ and 13 , we recover exactly the result of L. J. Mordell. The original example of Sir P. Swinnerton-Dyer reappears for $p = 7$, $a_1 = d_1 = a_2 = 1$, and $d_2 = 2$.

1.6. Remarks. — i) K/\mathbb{Q} is an abelian cubic field extension. It is totally ramified at p and unramified at all other primes. A prime $q \neq p$ is decomposed in K if and only if q is a cube modulo p .

ii) We will write \mathfrak{p} for the prime ideal in K lying above (p) . Note that $\mathfrak{p} = (\theta)$ by virtue of our definition of θ .

iii) We have $\prod_{i=1}^3 (T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2) = N_{K/\mathbb{Q}}(T_0 + \theta T_1 + \theta^2 T_2)$.

2 The proofs

2.1. Observations. — i) For s any solution of $T(a_1 + d_1T)(a_2 + d_2T) - 1 = 0$, the expression $\frac{a_1 + d_1s}{s}$ is well defined and non-zero in \mathbb{F}_p .

ii) No \mathbb{Q}_p -valued point on X reduces to the triple line “ $T_0 = T_3 = 0$ ”.

iii) For every $(t_0 : t_1 : t_2 : t_3) \in X(\mathbb{Q}_p)$, the fraction $\frac{a_1t_0 + d_1t_3}{t_3}$ is a p -adic unit.

Proof. i) By assumption, we have $s \neq 0$ and $a_1 + d_1s \neq 0$.

ii) Suppose, $(t_0 : t_1 : t_2 : t_3) \in X(\mathbb{Q}_p)$ is a point reducing to the triple line. We may assume $t_0, t_1, t_2, t_3 \in \mathbb{Z}_p$ are coprime. Then $\nu_p(t_0) \geq 1$ and $\nu_p(t_3) \geq 1$ together imply that $\nu_p(t_3(a_1t_0 + d_1t_3)(a_2t_0 + d_2t_3)) \geq 3$. On the other hand, $\nu_p(\prod_{i=1}^3 (t_0 + \theta^{(i)}t_1 + (\theta^{(i)})^2t_2)) = 1$ or 2 since t_1 or t_2 is a unit and $\nu_p(\theta^{(i)}) = \frac{1}{3}$.

iii) Again, assume $t_0, t_1, t_2, t_3 \in \mathbb{Z}_p$ to be coprime. Assertion ii) implies that t_3 is a p -adic unit. Hence, $\frac{a_1 t_0 + d_1 t_3}{t_3} \in \mathbb{Z}_p$. Further, $(\frac{a_1 t_0 + d_1 t_3}{t_3} \bmod p) = \frac{a_1 + d_1 s}{s}$ for $s := (t_3/t_0 \bmod p)$ a solution of $T(a_1 + d_1 T)(a_2 + d_2 T) - 1 = 0$. \square

2.2. Lemma. — *Let X be as in Theorem 1.4.ii) and $(t_0 : t_1 : t_2 : t_3) \in X(\mathbb{Q}_\nu)$. Further, let ν be any valuation of \mathbb{Q} different from ν_p and w an extension of ν to K .*

Then, $\frac{a_1 t_0 + d_1 t_3}{t_3} \in \mathbb{Q}_\nu^$ is in the image of the norm map $N : K_w \rightarrow \mathbb{Q}_\nu$.*

Proof. *First step:* Elementary cases.

If q is a prime decomposed in K then every element of \mathbb{Q}_q^* is a norm. The same applies to the infinite prime.

Second step: Preparations.

It remains to consider the case that q remains prime in K . Then, an element $x \in \mathbb{Q}_q^*$ is a norm if and only if $3 \mid \nu(x)$ for $\nu := \nu_q$.

It might happen that θ is not a unit in K_w . However, as K_w/\mathbb{Q}_q is unramified, there exists some $t \in \mathbb{Q}_q^*$ such that $\underline{\theta} := t\theta \in K_w$ is a unit. The surface \tilde{X} given by

$$T_3(a_1 T_0 + d_1 T_3)(a_2 T_0 + d_2 T_3) = \prod_{i=1}^3 (T_0 + \underline{\theta}^{(i)} T_1 + (\underline{\theta}^{(i)})^2 T_2)$$

is isomorphic to $X \times_{\text{Spec } \mathbb{Q}} \text{Spec } \mathbb{Q}_q$. Even more, the map

$$\iota : X \times_{\text{Spec } \mathbb{Q}} \text{Spec } \mathbb{Q}_q \rightarrow \tilde{X}, \quad (t_0 : t_1 : t_2 : t_3) \mapsto (t_0 : \frac{t_1}{t} : \frac{t_2}{t^2} : t_3)$$

is an isomorphism which leaves the rational function $\frac{a_1 T_0 + d_1 T_3}{T_3}$ unchanged. Hence, we may assume without restriction that $\theta \in K_w$ is a unit.

Third step: The case that θ is a unit.

Assume that $t_0, t_1, t_2, t_3 \in \mathbb{Z}_q$ are coprime. If $\nu(t_3(a_1 t_0 + d_1 t_3)(a_2 t_0 + d_2 t_3)) = 0$ then $\frac{a_1 t_0 + d_1 t_3}{t_3}$ is clearly a norm. Otherwise, we have $\nu(\prod_{i=1}^3 (t_0 + \theta^{(i)} t_1 + (\theta^{(i)})^2 t_2)) > 0$. This implies that $\nu(t_0), \nu(t_1), \nu(t_2) > 0$. Consequently, t_3 must be a unit.

From the equation of X , we deduce $\nu(d_1 d_2) > 0$. If $\nu(d_2) > 0$ then, according to the assumption, d_1 is a unit. This shows $\nu(a_1 t_0 + d_1 t_3) = 0$ from which the assertion follows.

Thus, assume $\nu(d_1) > 0$. Then, d_2 is a unit and, therefore, $\nu(a_2 t_0 + d_2 t_3) = 0$. Further, we note that

$$3 \mid \nu\left(\prod_{i=1}^3 (t_0 + \theta^{(i)} t_1 + (\theta^{(i)})^2 t_2)\right)$$

since the product is a norm. By consequence, $3 \mid \nu(t_3(a_1 t_0 + d_1 t_3)(a_2 t_0 + d_2 t_3))$. Altogether, we see that $3 \mid \nu(a_1 t_0 + d_1 t_3)$ and $3 \mid \nu(\frac{a_1 t_0 + d_1 t_3}{t_3})$. The assertion follows. \square

2.3. Proof of Theorem 1.4.ii). — According to Lemma 2.2, $\frac{a_1 t_0 + d_1 t_3}{t_3} \in \mathbb{Q}^*$ is a local norm at every prime except p . Global class field theory [Ta, Theorem 5.1 together with 6.3] shows that it must necessarily be a norm at that prime, too.

$(p) = \mathfrak{p}^3$ is a totally ramified prime. A p -adic unit u is a norm if and only if $\bar{u} := (u \bmod p)$ is a cube in \mathbb{F}_p^* . By Observation 2.1.iii), $\frac{a_1 t_0 + d_1 t_3}{t_3}$ is automatically a p -adic unit. As $\frac{a_1 + d_1 s}{s} = (\frac{a_1 t_0 + d_1 t_3}{t_3} \bmod p)$, this is exactly the assertion. \square

2.4. Proof of Theorem 1.4.iii). — We have to show that $X(\mathbb{Q}_\nu) \neq \emptyset$ for every valuation of \mathbb{Q} . $X(\mathbb{R}) \neq \emptyset$ is obvious. For a prime number q , in order to prove $X(\mathbb{Q}_q) \neq \emptyset$, we use Hensel's lemma. It is sufficient to verify that the reduction X_q has a smooth \mathbb{F}_q -valued point. Thereby, we may replace X by a \mathbb{Q}_q -scheme \tilde{X} isomorphic to $X \times_{\text{Spec } \mathbb{Q}} \text{Spec } \mathbb{Q}_q$.

Case 1: $q = p$.

Then, the reduction X_p is the union of three planes meeting in the line given by $T_0 = T_3 = 0$. By assumption, one of the planes appears with multiplicity one and is defined over \mathbb{F}_p . It contains p^2 smooth points.

Case 2: $q \neq p$.

Assume without restriction that θ is a w -adic unit. There are two subcases.

a) $q \nmid d_1 d_2$. It suffices to show that there is a smooth \mathbb{F}_q -valued point on the intersection X'_q of X_q with the hyperplane “ $T_0 = 0$ ”. This curve is given by

$$\bar{d}_1 \bar{d}_2 T_3^3 = \bar{\theta}^{(1)} \bar{\theta}^{(2)} \bar{\theta}^{(3)} \prod_{i=1}^3 (T_1 + \bar{\theta}^{(i)} T_2).$$

If $q \neq 3$ then this equation defines a smooth genus one curve. It has an \mathbb{F}_q -valued point by Hasse's bound.

If $q = 3$ then the projection $X'_q \rightarrow \mathbf{P}^1$ given by $(T_1 : T_2 : T_3) \mapsto (T_1 : T_2)$ is one-to-one on \mathbb{F}_q -valued points. At least one of them is smooth since $\prod_{i=1}^3 (T + \bar{\theta}^{(i)})$ is a separable polynomial.

b) $q \mid d_1 d_2$. $X'_q := X_q \cap “T_0 = 0”$ is given by $0 = \bar{\theta}^{(1)} \bar{\theta}^{(2)} \bar{\theta}^{(3)} \prod_{i=1}^3 (T_1 + \bar{\theta}^{(i)} T_2)$. In particular, $x = (0 : 0 : 0 : 1) \in X_q(\mathbb{F}_q)$. We may assume that x is singular.

Then, X_q is given as $Q(T_0, T_1, T_2)T_3 + K(T_0, T_1, T_2) = 0$ for Q a quadratic form and K a cubic form. If $Q \not\equiv 0$ then there is an \mathbb{F}_q -rational line ℓ through x such that $Q|_\ell \neq 0$. Hence, ℓ meets X_q twice in x and once in another \mathbb{F}_q -valued point which is smooth.

Otherwise, $(F \bmod q)$ does not depend on T_3 . I.e., the left hand side of the equation of X vanishes modulo q . This means that one of the factors must vanish. We have, say, $a_1 \equiv d_1 \equiv 0 \pmod{q}$. Then, by assumption, q decomposes completely in K . At such a prime, X'_q is the union of three lines which are defined over \mathbb{F}_q , different from each other, and meet in one point. There are plenty of smooth points on X'_q . \square

3 Examples

3.1. Example. — For $p = 19$, a counterexample to the Hasse principle is given by

$$\begin{aligned} T_3(19T_0 + 5T_3)(19T_0 + 4T_3) &= \prod_{i=1}^3 (T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2) \\ &= T_0^3 - 19T_0^2T_1 + 133T_0^2T_2 + 114T_0T_1^2 \\ &\quad - 1539T_0T_1T_2 + 5054T_0T_2^2 - 209T_1^3 \\ &\quad + 3971T_1^2T_2 - 23826T_1T_2^2 + 43681T_2^3. \end{aligned}$$

Indeed, in \mathbb{F}_{19} , the cubic equation $T^3 - 1 = 0$ has the three solutions 1, 7, and 11. However, in any case $\frac{a_1+d_1s}{s} = 5$ which is a non-cube.

3.2. Example. — Put $p = 19$. Consider the cubic surface X given by

$$T_3(T_0 + T_3)(12T_0 + T_3) = \prod_{i=1}^3 (T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2).$$

Then, $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$. X violates the Hasse principle.

Indeed, in \mathbb{F}_{19} , the cubic equation $T(1+T)(12+T) - 1 = 0$ has the three solutions 12, 15, and 17. However, in \mathbb{F}_{19} , $13/12 = 9$, $16/15 = 15$, and $18/17 = 10$ which are three non-cubes.

3.3. Example. — For $p = 19$, consider the cubic surface X given by

$$T_3(T_0 + T_3)(2T_0 + T_3) = \prod_{i=1}^3 (T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2).$$

Then, for X , the Hasse principle fails.

Indeed, in \mathbb{F}_{19} , the cubic equation $T(1+T)(2+T) - 1 = 0$ has $T = 5$ as its only solution. The two other solutions are conjugate to each other in \mathbb{F}_{19^2} . However, in \mathbb{F}_{19} , $6/5 = 5$ is a non-cube.

3.4. Example. — Put $p = 19$ and consider the cubic surface X given by

$$T_3(T_0 + T_3)(6T_0 + T_3) = \prod_{i=1}^3 (T_0 + \theta^{(i)}T_1 + (\theta^{(i)})^2T_2).$$

There are \mathbb{Q} -rational points on X but weak approximation fails.

Indeed, in \mathbb{F}_{19} , the cubic equation $T(1+T)(6+T) - 1 = 0$ has the three solutions 8, 9, and 14. However, in \mathbb{F}_{19} , $10/9 = 18$ is a cube while $9/8 = 13$ and $15/14 = 16$ are non-cubes. The smallest \mathbb{Q} -rational point on X is $(14 : 15 : 2 : (-7))$. Observe that, in fact, $T_3/T_0 = -7/14 \equiv 9 \pmod{19}$.

3.5. Remark. — From each of the examples given, by adding multiples of p to the coefficients a_1 , d_1 , a_2 , and d_2 , a family of surfaces arises which are of similar nature.

3.6. Remark (Lattice basis reduction). — The norm form in the $p = 19$ examples produces coefficients which are rather large. An equivalent form with smaller coefficients may be obtained using lattice basis reduction. In its simplest form, this means the following.

For the rank 2 lattice in \mathbb{R}^3 , generated by the vectors $v_1 := (\theta^{(1)}, \theta^{(2)}, \theta^{(3)})$ and $v_2 := ((\theta^{(1)})^2, (\theta^{(2)})^2, (\theta^{(3)})^2)$, in fact $\{v_1, v_2 + 7v_1\}$ is a reduced basis. Therefore, the substitution $T'_1 := T_1 - 7T_2$ simplifies the norm form. Actually, we find

$$\prod_{i=1}^3 (T_0 + \theta^{(i)} T_1 + (\theta^{(i)})^2 T_2) = T_0^3 - 19T_0^2 T'_1 + 114T_0 T_1'^2 + 57T_0 T'_1 T_2 - 133T_0 T_2^2 \\ - 209T_1'^3 - 418T_1'^2 T_2 + 1045T'_1 T_2^2 - 209T_2^3.$$

References

- [Ma] Manin, Yu. I.: *Cubic forms, algebra, geometry, arithmetic*, North-Holland Publishing Co. and American Elsevier Publishing Co., Amsterdam-London and New York 1974
- [Mo] Mordell, L. J.: *On the conjecture for the rational points on a cubic surface*, J. London Math. Soc. **40** (1965), 149–158
- [SD] Swinnerton-Dyer, Sir P.: *Two special cubic surfaces*, Mathematika **9** (1962), 54–56
- [Ta] Tate, J.: *Global class field theory*, in: Algebraic number theory, Edited by J. W. S. Cassels and A. Fröhlich, Academic Press and Thompson Book Co., London and Washington 1967